MMM Competition 2025

Problem 1.

We call a set *sum-free* if no two elements in the set add up to a third element in the set. Consider the set $\{1,2\}$. This set has four subsets: the emptyset $\emptyset$, the singletons $\{1\}$, $\{2\}$ and the set itself $\{1,2\}$. Out of these four subsets, $\emptyset$, $\{1\}$ and $\{2\}$ are sum-free. For example, $\{1\}$ is sum-free, because $1+1=2$ and $2 \notin \{1\}$. On the other hand $\{1,2\}$ is not sum-free, because $1+1=2$ and $2 \in \{1,2\}$.

We call a set *product-free* if no two elements in the set multiply to a third element in the set. For example, $\{1\}$ is not product-free, because $1 \cdot 1 = 1$ and $1 \in \{1\}$.

(a) What is the number of sum-free subsets of $\{1,2,3,4,5,6\}$?

(b) Give an example of a subset of $\{1,2,3,4,5,6\}$ that is product-free and larger in size than any sum-free subset of $\{1,2,3,4,5,6\}$.

(c) Let $N = \{1,2,\ldots,n\}$ be the set of first $n \geq 2$ integers. What is the maximum size of a sum-free subset of N as a function of n?

(d) Give an example of a maximum size sum-free subset of N that is also product-free.

Problem 2.

In each of the following three games, players take turns. That is, player 1 starts, then player 2, then player 1, and so on. For each of the following three games, determine which player has a winning strategy: player 1, player 2, or neither. If there is a player with a winning strategy, then describe the winning strategy.

Game 1

Player 1 starts by saying the number 1, then player 2 says the number 2, and the one who's next must say an integer strictly between the previous number and twice of it (not including the endpoints).

For example, player 1 starts by saying 1, then player 2 says 2. Player 1's options are now all integers strictly between 2 and 4, but there's only one such option: 3. So player 1 says 3. Player 2's options are now between 3 and 6, which are 4 and 5.

The game finishes when someone says 100 or larger; that player wins.

Game 2

Consider a row of $n \geq 2$ coins of values $v_1, \ldots, v_n$, where n is even. Assume that $v_1 + v_2 + \ldots + v_n$ is odd.

For example:

7 9 4 3 3 5

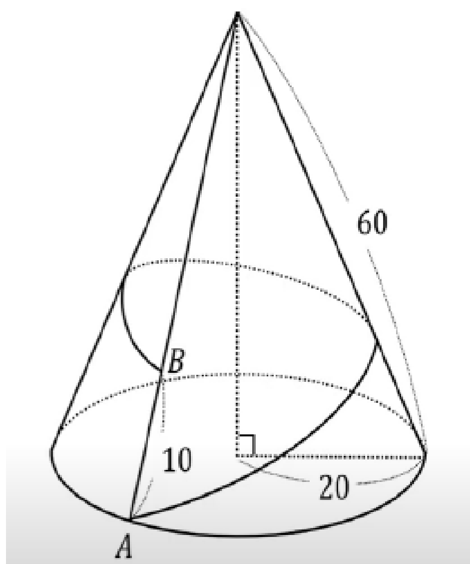
In each turn, a player selects either the first or last coin from the row, removes it from the row, and receives the value of the coin. The game finishes when all coins are selected. The player with the largest value of selected coins wins.

Game 3

There are nine cards numbered from 1 through 9. In each turn, a player selects a card. The first player to have three cards that add up to 15 wins. If after picking all nine cards, no player has three cards that add up to 15, the game ends in a draw.

Problem 3.

The diagram below illustrates a cone shaped mountain with a circular base with radius 20 and length of the sides equal to 60. The plan is to build a track for a sightseeing train around the mountain, which starts at A, at the bottom of the mountain, and ends up at B, which is exactly 10 away from A in the uphill direction. The track goes around the mountain exactly once, and should be covering the shortest possible distance.



(a) What is the length of this track?

(b) Looking at the picture carefully, you can see that a part of this track actually goes downhill! What is the length of this part of the track (the part that goes downhill)?

Problem 4.

The following conversation took place between two friends Alice and Bob.

Alice: I have a rectangular box with integer-valued side lengths a, b and c.

In the conversation between Alice and Bob the numbers a, b and c are of course told, but that would spoil the fun of the exercise. We do make the assumption that a is the smallest length and c is the biggest, so that $a \leq b \leq c$.

Bob: That's funny, I have a rectangular box too, but all the sides are exactly 2 longer than the sides of your box!

Alice: I notice another coincidence: The volume of your box is exactly double the volume of mine! How about that?

Bob: Neat! Now that I think about it: The value of c is actually the highest possible value of an integer that makes this possible!

(a) What is this value of c?

(b) Show that you can't find a higher value than you found in exercise (a).

Problem 5.

For the weekend, Alice bought three apple pies, to share with her visitors Bob and Carol.

(a) On Friday evening Bob is there, but Carol is stuck in the traffic. Having waited for two hours Carol still hasn't arrived. So Alice and Bob decide they go ahead with the first apple pie anyway. First Alice takes half of the apple pie. Then Bob takes half of the half that remains. Then Alice takes half of what is still left. Then Bob takes half of what is left. And so they continue, on and on, taking turns and always taking half of what is still left. In the long run, how much of the apple pie do Alice and Bob each get?

(b) On Saturday afternoon, the second apple pie is shared. Carol has finally arrived and this pie is shared by the three of them. Again they take turns: first Alice, then Bob, then Carol, then Alice, then Bob, then Carol, and so on. Alice always takes one third of the part of the pie that is still left. Bob always takes part "b" of what is left. Carol always takes one fifth of what is left. In the long run, it turns out that the total amount that Bob is getting, happens to also equal part "b" of the original pie.

Knowing that $0 < b < 1$, what is the value of "b"?

(c) On Sunday morning, the third apple pie is shared. Again Alice, Bob, and Carol are taking turns: A, B, C, A, B, C, A, B, C, ... Alice always takes part "a" of what is still left, Bob always takes part "b", and Carol takes part "c". This time $a+b+c=1$, and it happens that in the long run Alice gets in total part "c" of the original pie, Bob gets in total part "b", and Carol gets in total part "a".

Knowing that $0 < a < 1$, $0 < b < 1$ and $0 < c < 1$, what are the values of "a", "b" and "c"?