MMM Competition 2025

Problem 1.

Solution.

(a) 24: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$,$\{6\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{2,3\}$, $\{2,5\}$, $\{2,6\}$, $\{3,4\}$, $\{3,5\}$, $\{4,5\}$, $\{4,6\}$, $\{5,6\}$, $\{1,3,5\}$, $\{1,4,6\}$, $\{2,5,6\}$, $\{3,4,5\}$, $\{4,5,6\}$

(b) $\{3,4,5,6\}$

(c) $n/2$ if n is even, and $(n+1)/2$ is n is odd

(d) $\{\lfloor n/2 \rfloor + 1, \ldots, n\}$

Problem 2.

Game 1

Solution. Player 1 can guarantee a win:

Round	Player 1	Player 2
1	1	
2		2
3	3	
4		4,5
5	6	
6		7,8,9,10,11
7	12	
8		$13 \leq x \leq 23$
9	25	
10		$26 \leq x \leq 49$
11	50	
12		$51 \leq x \leq 99$
13	100	

Game 2

Solution. Player 1 always wins by selecting the best of even and odd coins

Game 3

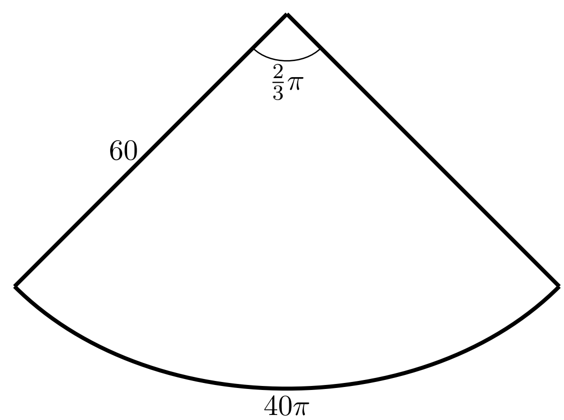
Solution. There are 8 ways to get 15: 1,5,9 or 1,6,8 or 2,4,9 or 2,5,8 or 2,6,7 or 3,4,8 or 3,5,7 or 3,4,8 or 4,5,6

8	1	6
3	5	7
4	9	2

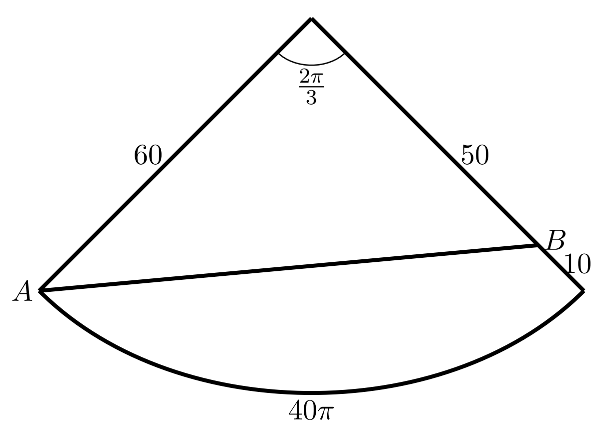
Game is equivalent to tic-tac-toe: no winning strategy

Problem 3.

Solution. We start by 'unwrapping' the cone. This gives us part of a circle. In this case that would be a circle with radius 60, where the arc length is the length of the circumference of the circle that formed the base of the cone. That length is $2 \cdot \pi \cdot R = 2 \cdot \pi \cdot 20 = 40\pi$. With the radius being 60 the length of the circumference of the entire circle would be 120π, so apparently we have $\frac{1}{3}$ of a circle, meaning that the angle at the peak is $\frac{2}{3}\pi$.

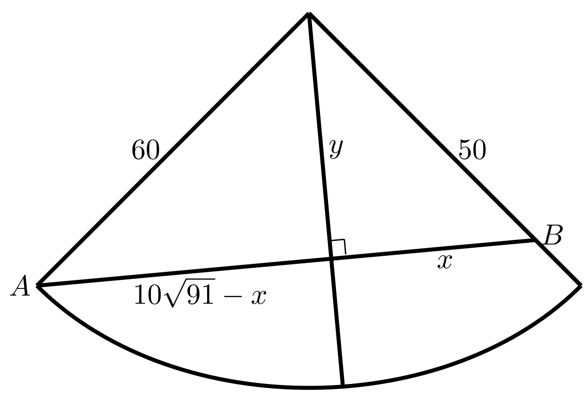


In this drawing A is located at the point on the left and B is on the edge on the right at distance 10 from the circle (and 50 from the peak). This gives:



The length AB can be calculated using the law of cosines: $AB^2 = 60^2 + 50^2 - 2 \cdot 50 \cdot 60 \cdot \cos(\frac{2}{3}\pi) = 6100 - 2 \cdot 3000 \cdot -\frac{1}{2} = 9100$, so the length of the track is $\sqrt{9100} = 10\sqrt{91} (= 95.3939)$.

For the part that goes downhill, notice that going downhill means that we get closer to the base (or further away from the peak). So we need to find the point on AB that is farthest away from the base. We find that point by drawing the line from the peak to the base that is perpendicular to the line AB:



Now we use the Pythagorean theorem twice to solve for x, which is the downhill part of the track. We get:

- $(10\sqrt{91} - x)^2 + y^2 = 60^2$
- $x^2 + y^2 = 50^2$

Subtracting the two equations cancels out the y^2 and also the x^2, leaving us with the following equation:

$$9100 - 20\sqrt{91}x = 1100$$

which gives $x = \frac{400}{\sqrt{91}} (= 41.9314)$

Problem 4.

Solution. We have two boxes with integer-valued side lengths a, b, c and $a+2$, $b+2$ and $c+2$ and the volume of the bigger box is twice the volume of the smaller box: $(a+2)(b+2)(c+2) = 2abc$.

Notice first that $a \geq 3$, because if $a = 2$, then $a+2 = 2a$ so the volume of the bigger box will be more than double the volume of the smaller box. Solving the equation $(a+2)(b+2)(c+2) = 2abc$ for c gives $[2ab-(a+2)(b+2)]c = 2(a+2)(b+2)$, so

$$c = \frac{2(a+2)(b+2)}{ab-2a-2b-4} = \frac{2(a+2)(b+2)}{(a-2)(b-2)-8}$$

In order to maximize this expression, we want to minimize the denominator: We try first values of a and b that make $(a-2)(b-2)-8 = 1$, or $(a-2)(b-2) = 9$. There are only two solutions of this equation, namely $(a,b) = (5,5)$ and $(a,b) = (3,11)$. $(a,b) = (5,5)$ gives $c = \frac{2\cdot 7\cdot 7}{3\cdot 3-1} = 98$ and $(a,b) = (3,11)$ gives $c = \frac{2\cdot 5\cdot 13}{1\cdot 9-8} = 130$, which is the solution to question a.

To show that we can't get a better solution than 130, let $x = a-2 \geq 1$ and $y = b-2 \geq 1$. Notice that in that case $0 \leq (x-1)(y-1) = xy-x-y+1$, so $x+y \leq xy+1$ and don't forget that $xy-8 = (a-2)(b-2)-8 \geq 1$. We have:

$$\begin{aligned}
c &= \frac{2(x+4)(y+4)}{xy-8} \\
&= \frac{2xy+8(x+y)+32}{xy-8} \\
&\leq \frac{2xy+8(xy+1)+32}{xy-8} \\
&= \frac{10xy-80+120}{xy-8} \\
&= 10+\frac{120}{xy-8} \leq 10+120 = 130
\end{aligned}$$

Problem 5.

Solution.

(a) Alice gets: $p_A = 1/2 + (1/2)^3 + (1/2)^5 + (1/2)^7 + \ldots$

Bob gets: $p_B = (1/2)^2 + (1/2)^4 + (1/2)^6 + (1/2)^8 + \ldots$

In these infinite sums you may recognize the "geometric series", for which there is the formula $1 + r + r^2 + r^3 + \ldots = 1/(1-r)$, for every r in the interval $-1 < r < 1$. Then: $p_A = 1/2(1 + r + r^2 + r^3 + \ldots)$ with $r = 1/4$, giving $p_A = 2/3$. Likewise, $p_B = (1/2)^2(1+r+r^2+r^3+\ldots) = 1/3$. Alternatively: note that in every round Bob gets half of what Alice gets. Therefore, Bob's total sum is also half of Alice's total sum. Together, in the long run the whole pie is shared, making $p_A + p_B = 1$. This gives the same answer, but faster.

(b) In the first round, Alice gets $1/3$, Bob gets $(1-1/3)b = (2/3)b$, and Carol gets $(1–1/3)(1-b)(1/5) = 2(1-b)/15$. In each later round they get the same individual proportions of the amount that was there at the beginning of the round. This means that the total amounts they collect have the same relative sizes: $1/3$, $2b/3$, and $2(1-b)/15$, but they add up to 1. Therefore: $p_A = M \cdot 1/3$, $p_B = M \cdot 2b/3$, and $p_C = M \cdot 2(1-b)/15$, for a suitable scaling factor M. With $p_B = b$ (and b non-zero) it follows that $M = 3/2$. Then: $1 = p_A + p_B + p_C = 1/2 + b + (1-b)/5 = 7/10 + 4b/5$, so $3/10 = 4b/5$. Therefore: $b = 3/8$.

(c) Following the same strategy as for question (b), we now find that: $p_A = M \cdot a$, $p_B = M \cdot (1-a)b$, and $p_C = M \cdot (1-a)(1-b)c$, for a suitable scaling factor M.

Also: $p_A = c$, $p_B = b$, and $p_C = a$. Therefore, comparing the expressions for p_B (and with b non-zero), we find: $M(1-a) = 1$, giving $M = 1/(1-a)$.

So: $p_A = a/(1-a) = c$, $p_C = (1-b)c = a$, while $a + b + c = 1$. From the first identity we have $a = (1-a)c$, while the second identity states $a = (1-b)c$. Therefore: $a = b$.

So we have the values: a, $b = a$, and $c = 1 - 2a$. To compute a, we consider $a = (1-a)(1-2a)$, which leads to $a = 1 - 3a + 2a^2$, and

$2a^2 - 4a + 1 = 0$. This is a quadratic equation which is easily solved to give $a = 1 + \sqrt{2}/2$ or $a = 1 - \sqrt{2}/2$. Only the negative sign gives a feasible solution, because $0 < a < 1$. It follows that there is a unique solution: $a = b = 1 - \sqrt{2}/2 = 0.292893$, $c = -1 + \sqrt{2} = 0.414213$.