MMM Competition 2024

Problem 1.

(a)

$$10 = 1 \times 10 = 2 \times 5 \Rightarrow 4 \text{ divisors, so white}$$
$$64 = 1 \times 64 = 2 \times 32 = 4 \times 16 = 8 \times 8 \Rightarrow 7 \text{ divisors, so black}$$
$$123 = 1 \times 123 = 3 \times 41 \Rightarrow 4 \text{ divisors, so white}$$

(b) Paintings with a square number are black and all other paintings white. So 11 black paintings: 1,4,9,16,25,36,49,64,81,100,121.

(c) We make us of prime factorizations.

$98 = 2^1 \cdot 7^2$. So there are $2 \cdot 3 = 6$ divisors, of which 3 are odd (the exponent related to 2 must be 0) and thus 3 even.

$120 = 2^3 \cdot 3^1 \cdot 5^1$. So there are $4 \cdot 2 \cdot 2 = 16$ divisors, of which $2 \cdot 2 = 4$ are odd and thus 12 even.

Problem 2.

(a)

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}} \Leftrightarrow \sqrt{2} - 1 = \frac{1}{1+\sqrt{2}} \Leftrightarrow (\sqrt{2}-1)(1+\sqrt{2}) = 1.$$

(b)

$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}} = 1 + \frac{1}{2 + \frac{1}{1+\sqrt{2}}} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{1+\sqrt{2}}}} = \dots$$

Hence $a_0 = 1$ and $a_1 = a_2 = a_3 = \dots = 2$.

(c) Yes. We consider two cases.

- If Bob has black hair: yes, because Bob sees Charly
- If Bob has blond hair: yes, because Alice sees Bob

(d) Consider $\sqrt{2}^{\sqrt{2}}$. We consider two cases:

- If $\sqrt{2}^{\sqrt{2}}$ is rational, then the statement is true by choosing $x = y = \sqrt{2}$.
- If $\sqrt{2}^{\sqrt{2}}$ is irrational, then the statement is true by choosing $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, because

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2.$$

Problem 3.

Notice first that Freddie's x-coordinate is irrelevant, so we only need to take his current y-coordinate into account to calculate how long it takes for him to reach the river. So let $E(y)$ be the expected value of the number of jumps it will take Freddy to reach the river from a given y-coordinate. We have:

$$
\begin{aligned}
E(24) &= 0 \\
E(0) &= 1 + \tfrac{2}{3}E(0) + \tfrac{1}{3}E(1) \\
E(y) &= 1 + \tfrac{1}{4}E(y+1) + \tfrac{1}{2}E(y) + \tfrac{1}{4}E(y-1),\ y = 1, 2, 3, \ldots, 23
\end{aligned}
$$

The second equation gives $E(0) = 3 + E(1)$. Plugging that in in the equation for $E(1)$ gives $E(1) = 1 + \frac{E(2)+3E(1)+3}{4}$, which results in $E(1) = 7 + E(2)$. If we continue this procedure, we find:

$$
\begin{aligned}
E(2) &= 11 + E(3) \\
E(3) &= 15 + E(4) \\
&\cdots\cdots \\
E(y) &= 4y + 3 + E(y+1) \\
&\cdots\cdots \\
E(23) &= 95 + E(24)
\end{aligned}
$$

Since $E(24) = 0$, we find that $E(23) = 95$, $E(22) = 95 + 91 = 186$ and, finally $E(21) = 186 + 87 = 273$.

Problem 4.

Say that the common ratio in the geometric progression is r and that a, b and c are the initial number of peanuts of Alice, Bob and Charlie respectively. Then: $a+b+c = 1397$ and $b = ar$ and $c = br = ar^2$. This gives $a\cdot(1+r+r^2) = 1397$, which can be rewritten as $1 + r + r^2 = \frac{1397}{a}$, or

$$(r+1)^2 = r + \tfrac{1397}{a}$$

Now let d be the common difference in the arithmetic progression. Then we have that $d = (b-9)-(a-5) = (c-24)-(b-9)$ (with $b = ar$ and $c = ar^2$). This gives: ($d =$) $ar - 9 - (a-5) = ar^2 - 24 - (ar - 9)$, which can be rewritten as $ar^2 - 2ar + a - 11 = 0$. This gives:

$$r^2 - 2r + 1 = (r-1)^2 = \tfrac{11}{a}$$

Subtracting the two displayed formulas gives $(r+1)^2 - (r-1)^2 = 4r = r + \frac{1397}{a} - \frac{11}{a}$, or

$$r = \tfrac{462}{a}$$

Plugging this in in the equation $a\cdot(1+r+r^2) = 1397$, results in the following equation in a only: $a\cdot(1 + \frac{462}{a} + \frac{462^2}{a^2}) = 1397$, which reduces to $a^2 - 935a + 462^2 = 0$, which we can solve using the quadratic equation:

$$a = \tfrac{1}{2}\cdot(935 \pm \sqrt{935^2 - 4\cdot 462^2}) = \tfrac{1}{2}(935 \pm 143)$$

which gives $a = 396$ or $a = 539$. Since a is the smallest one of the three numbers a, b and c, it must be smaller than $\frac{1397}{3} = 465\frac{2}{3}$, so $a = 397$. The common ratio r then equals $\frac{462}{a} = \frac{462}{397} = \frac{7}{6}$.

Problem 5.

QUESTION A (2 points)
What are all the **possible outcomes** of this procedure? (Explain.)

By inserting subtraction first, we can always group the terms in the expression as follows: the first with the second, the third with the fourth, and so on.
For the selection in the example: (24 - 21) + (20 - 18) + (17 - 15) + (14 - 13) + (8 - 7) + (6 - 3).
In case of an odd number of numbers, the last number is simply a group on its own. Each of the outcomes for the groups in this expression is strictly positive, because the sequence is strictly monotonically decreasing, therefore the overall outcome is strictly positive (unless all numbers were removed, then it is 0). If there are two consecutive groups, a larger result is obtained by "merging them", by deleting the adjoining numbers in the middle, e.g.: if (24 - 21) + (20 - 18) is changed into (24 - 18), the result is larger because – 21 + 20 = -1 is strictly negative and gets removed in the middle. In this way we find that the maximum outcome is 24. Since each individual number could be kept (and the other 23 removed), the possible outcomes are: 0, 1, 2, …, 24.

QUESTION B (2 points)
How many different expressions can you create in this way? (Explain.)

Each number is either kept or removed. Once these choices are made, the rest of the procedure (inserting subtractions and additions and computing the outcome) is completely determined. Therefore, we have to make 24 binary choices, implying that there are 2^24 = 16,777,216 different expressions.

QUESTION C (6 points)
Following the rules above, in **how many ways** can you create an expression which **equals 3**? (Show all your work.)

There are two ways you can go about it.

(1) The hard way, by systematically analyzing in which way the outcome three can be obtained from such expressions, step by step. Not too difficult, but easy to make mistakes. For instance, in the following manner:
* When using 0 operations, there is 1 way to get the outcome 3 from a single number.
* When using 1 operation, there are 21 ways to arrive at the outcome 3: from 24-21, 23-20, 22-19, …, 4-1.
* When using 2 operations, there are 42 ways to arrive at the outcome 3. The first operation is subtraction, producing an intermediate value that later can only become larger. So it must be either 1 or 2. The second operation adds a positive number achieving the outcome 3. Hence it is either 2 or 1. Therefore we have: 24-23+2, 23-22+2, …, 4-3+2 (21 options) and

24-22+1, 23-21+1, ..., 4-2+1 (another 21 options).
* When using 3 operations, there are 420 ways to get at the outcome 3. We then get an expression of the form (a-b)+(c-d) with a>b>c>d. To have the outcome 3, this effectively reduces to either a-b=1 and c-d=2, or a-b=2 and c-d=1.
We have: 24-23+22-20, 24-23+21-19, 24-23+20-18, ..., 24-23+3-1 (20 options),
23-22+21-19, 23-22+20-18, ..., 23-22+3-1 (19 options),
....
5-4+3-1 (1 option).
And likewise: 24-22+21-20, 24-22+20-19, 24-22+19-18, ..., 24-22+2-1 (20 options),
23-21+20-19, 23-21+19-18, ..., 23-21+2-1 (19 options),
....
5-3+2-1 (1 option).
In total there are two times 20+19+...+1 options, which makes 2* 20*21/2 = 420.
* When using 4 operations, there are 210 ways to get to the outcome 3. In this case the expression takes the form (a-b)+(c-d)+e with a-b=1, c-d=1 and e=1. Also: a>b>c>d>e.
Now we must count: 24-23+22-21+1, 24-23+21-20+1, ..., 24-23+3-2+1 (20 options).
Next: 23-22+21-20+1, 23-22+20-19+1, ..., 23-22+3-2+1 (19 options).
...
5-4+3-2+1 (1 option).
In total there are 20+19+...+1 options, which makes 20*21/2 = 210.
* When using 5 operations, there are 1330 ways to get to the outcome 3.
Now we have an expression of the form (a-b)+(c-d)+(e-f), with a-b=1, c-d=1 and e-f=1.
Therefore:
24-23+22-21+20-19, 24-23+22-21+19-18, ..., 24-23+22-21+2-1, (19 options)
24-23+21-20+19-18, 24-23+21-20+18-17, ..., 24-23+21-20+2-1, (18 options)
....
24-23+4-3+2-1, (1 option)
23-22+21-20+19-18, 23-22+21-20+18-17, ..., 23-22+21-20+2-1, (18 options)
23-22+20-19+18-17, 23-22+20-19+17-16, ..., 23-22+20-19+2-1, (17 options)
...
23-22+4-3+2-1, (1 option)
...
...
6-5+4-3+2-1. (1 option)
This adds up to: (19+18+...+1) + (18+17+...+1) + ... + (3+2+1) + (2+1) + 1 = 1330.

* In total, this gives: 1 + 21 + 42 + 420 + 210 + 1330 = 2024 ways.

(2) **The clever way**, by figuring out the general pattern starting from much smaller numbers and proving this by induction.

Suppose we have the list of n numbers in decreasing order, and we would like to know in how many ways the final outcome can be made to equal k.

* For n=1, the only possible outcomes are k=0 and k=1. Each can happen in precisely 1 way.
* For n=2, the possible outcomes are k=0, k=1 and k=2. We have: k=0 is only obtained in 1 way, k=1 is obtained in 2 ways (2-1 and just 1), k=2 is obtained in 1 way (just 2).

* For n=3:
k=0 happens in 1 way (all numbers removed).
k=1 happens in 3 ways (expressions: 3-2, 2-1, 1).
k=2 happens in 3 ways (expressions: 3-2+1, 3-1, 2).
k=3 happens in 1 way (expression 3).

What we may recognize are the numbers that appear in the Pascal Triangle (also known as the Newton Binomium).

n=0					1				
n=1				1		1			
n=2			1		2		1		
n=3		1		3		3		1	
n=4	1		4		6		4		1

It now remains to prove the hypothesis that these numbers indeed occur **for arbitrary n and k**. We do so with induction, assuming this result to hold up to the value n-1. If now we have n numbers available and the outcome of an expression equals k, this expression either *does not* have the number n to start with (it was removed), or it *does*.
In the first case, we are effectively having an expression with outcome k for only the numbers up to n-1. By the induction hypothesis that we are dealing with the binomial coefficients, this equals "n-1 choose k".
In the second case, the number n is present in the expression and followed by an expression of the form – a + b – c +….. which can be written as – (a – b + c - …) with the numbers a>b>c… all less or equal to n-1 and the expression in parentheses equal to n-k (to make sure that the final outcome indeed equals k). By the induction hypothesis for n-1, this number then equals the binomial coefficient "n-1 choose n-k". Because of symmetry this is also equal to "n-1 choose n-1-(n-k)" or "n-1 choose k-1".
Adding them up, we find "n-1 choose k" + "n-1 choose k-1" which is well known to be equal to "n choose k" (this is how you build the next rows in the triangle), proving the result.
Finally, for n=24 and k=3 we get the answer "24 choose 3" which equals 2024.