MMM Competition 2023

Solution Problem 1.

(a) You.

(b) Eat three cookies in the first round. In all following rounds, if your brother eats $c = 1, 2, 3$ cookies, you eat $4 - c$ cookies.

(c) $n \neq 1, 5, 9, 13, 17, \ldots$, or equivalently $n \neq 1 + 4k$ for some $k \in \mathbb{N}$.

Solution Problem 2.

(a) 17.

Order is 9,7,2,14,11,5,4,12,13,3,6,10,15,1,8, or the reverse.

(b) 18.

For $n = 16$: Order is 16, 9,7,2,14,11,5,4,12,13,3,6,10,15,1,8, or the reverse.

For $n = 17$: Order is 16, 9,7,2,14,11,5,4,12,13,3,6,10,15,1,8, 17, or the reverse.

For $n = 18$: 16, 17 and 18 have a unique neighbor.

Solution Problem 3.

(a) 7 red marbles means that we can have at most 14 times that the right hand neighbor of a marble is one of the other color (namely if we have an arrangement that has the 'subsequence' BRB in there 7 times; all other marbles must then be blue). This means that we also need to have 15 times the subsequence BB (in order to have 14 marbles whose right hand neighbor is the same color). We get that by adding 14 additional blue marbles to the sequence $BRBRBRBRBRBRBRB$, giving us a total of $14 + 8 = 22$ blue marbles. So $m = 22$.

(b) This one we can solve by first considering the $BRBRBRBRBRBRBRB$ arrangement. We start of with this arrangement in order to make sure that we get 14 times that right hand neighbor of a marble is one of the other color. And then we add the other 14 blue marbles. Each time when we add a blue marble it is placed in one of 8 places, where a place is the number of red marbles to its left. (Note: this number can be 0, 1, 2, 3, 4, 5, 6 or 7). This means that we have a combinatorics problem here where we do 14 selections out of a group of size 8 where the selections unordered (because the blue marbles are identical) and repetition is allowed (in the sense that we can select the same 'number of red marbles to its left' multiple times). If we, in such a situation, make k selections from a group of size n, the number of ways to do so is $\binom{n+k-1}{k}$. In this case $n = 8$ and $k = 14$, giving us $\binom{21}{14} = 116280$ arrangements.

Solution Problem 4.

(a) $a_n = 1000 + n^2$

(b) We make use of a few tricks to calculate this GCD:

- $GCD(x, y) = GCD(x - y, y)$
- If y and a don't share any common factors, then $GCD(ax, y) = GCD(x, y)$

We then proceed to essentially perform Euclid's Algorithm:

$$\begin{aligned}
GCD(a_n, a_n + 1) &= GCD(1000 + n^2, 1000 + (n+1)^2) \\
&= GCD(n^2 + 1000, n^2 + 2n + 1001) \\
&= GCD(n^2 + 1000, 2n + 1) \\
&= GCD(4n^2 + 4000, 2n + 1) \\
&= GCD(4n^2 + 4000 - (2n+1)^2, 2n + 1) \\
&= GCD(-4n + 3999, 2n + 1) \\
&= GCD(4001, 2n + 1) \\
&\leq 4001
\end{aligned}$$

and by choosing $n = 2000$ (or $n = 6001$ or ...) we can make that GCD equal to 4001. So that is the answer.

Solution Problem 5.

(a) (i)

(b) (iii)

(c) (iii)

(d) (ii)

(e) (ii)

What do you do if you don't have a clue? - SOLUTIONS

You gamble! Below are five multiple choice questions on optimality. You don't need to provide any computation or argumentation.

(a) If you connect locations in a flat country like the Netherlands with a railroad network of minimal total length, then any three railroad tracks meeting somewhere in the network always make angles of 120 degrees.
Sketched are three different ways of connecting 6 places located in the corners of a regular hexagon by a network that meets this optimality criterion.

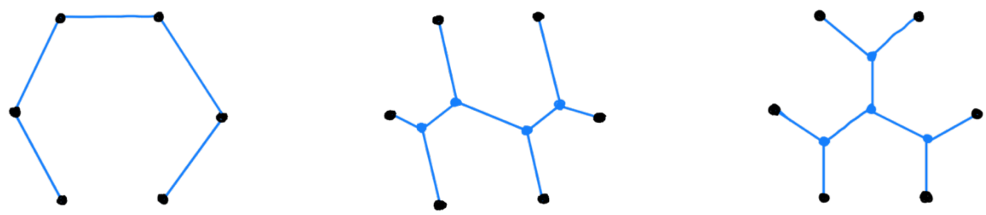


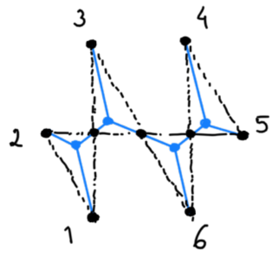
Figure 1 Figure 2 Figure 3

Which one has **minimal total length**?

(i) Figure 1
(ii) Figure 2
(iii) Figure 3

SOLUTION (a)

The property mentioned in this exercise, refers to the Fermat point of a triangle, which minimizes the sum of the distances of an interior point to the corners. It has the property that the three lines from the corners of the triangle to the Fermat point meet there under angles of 120 degrees only.
Let the distances between the corners of the hexagon all be normalized as 1.
Figure 1 is easy. It has a network of total distance 5.
Figure 2 has a network of total distance $2\sqrt{7} = 5.29150$. This is the hardest figure to analyze. Note by symmetry that the network passes through the center of the hexagon. If we label the corner points from 1 to 6 starting at the left bottom corner, we can split the network into 4 congruent parts as follows:

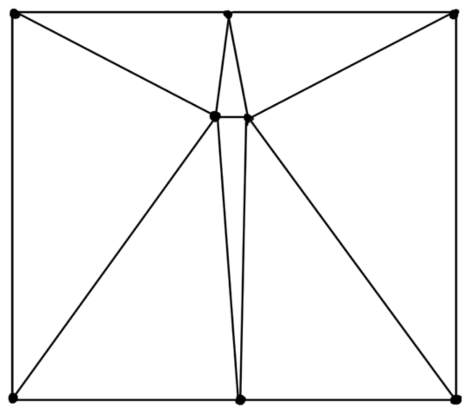


(part 1) corners 1 and 2, and the middle point between the first two blue points; (part 2) corner 3, the previous middle point, and the center of the hexagon; (part 3) corner 6, the center of the hexagon, and the middle point between the last two blue points; (part 4) corners 4 and 5 and the previous middle point. Each of these four triangles is a rectangular triangle with sides of length 1, ½, and ½√3. This allows us to compute the Fermat point inside, and the total length of the network inside such a triangle as ½√7. Overall, this gives 2√7.

Figure 3 has equilateral triangles, and a network of total distance 3√3 = 5.19615.
Figure 1, the least spectacular network, has the shortest total distance.
Answer (i): Figure 1.

(b) Today, I was a bit clumsy, and I dropped a square glass mirror which broke into several sharp pieces when it hit the ground. Then I noticed that all the pieces happened to be triangles with acute angles only (so, all angles were strictly less than 90 degrees).
Into **how many pieces** did the mirror break **at least**?
(i) 6
(ii) 7
(iii) 8
(iv) 9

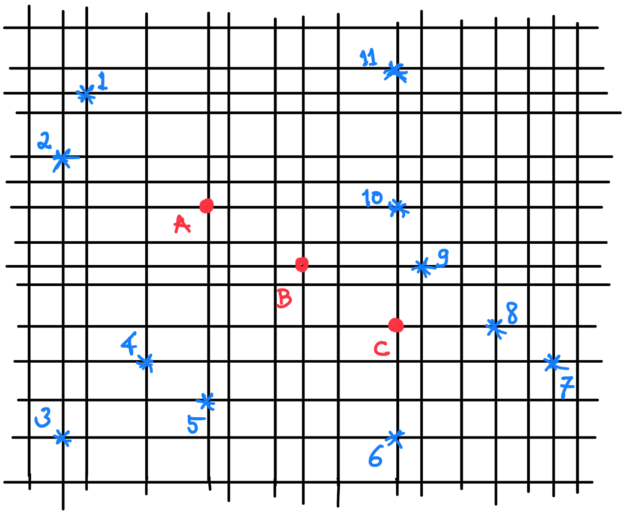
SOLUTION (b)
Several observations may help you build some intuition to find a solution. (1) Each of the four corners of the mirror cannot remain intact, as they are originally 90 degrees and must be reduced to get acute angles only. (2) For similar reasons, every point on the edge of the mirror where breaking lines meet, must have at least 3 lines meeting. (3) And every interior point inside the square where breaking lines meet, must either have at least 5 lines meeting there, or two of them from a continued straight line which is the edge of a triangular piece and at least three others land there from one side only. You will quickly discover that you must have at least two interior points. And you will also need some edge points.
Here is a possible solution:



Answer (iii): at least 8 pieces are needed.
On the internet you can find a full proof. Also, you can find that 9 such pieces are in fact impossible, but 10 or more are.

(c) The Council of Eleven (de Raod vaan èlf) of the Maastricht Carnival Association "De Tempeleers" went on a business trip to New York. They all booked different hotels in a part of Manhattan where the roads form a rectangular grid of Streets and Avenues.

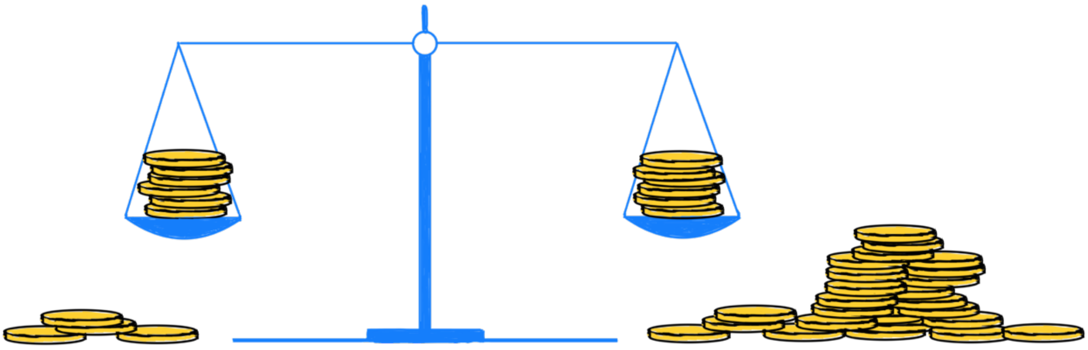
Below is a map of their locations. They would like to meet at a convenient location in town, such that **the total distance** they jointly must walk to get there is **minimal**.



What is the best location for them to meet?
(i) Location A
(ii) Location B
(iii) Location C

SOLUTION (c)
Suppose all 11 council members ended up in the same Street (a one-dimensional situation). Then, clearly, they would be looking for a best place along that same street. (Otherwise, any other meeting point could be projected onto the street to give a better result.) Starting on one end of the street and shifting slowly along it, you would see that such shifting decreases the total distance only if we shift away from the street's end with few councilmembers and towards the end with many council members. This works until there are equally many council members on each side of the point along the street: you should go for **the median** of the locations.
For the 2D-situation of Manhattan, every distance is the sum of the vertical and the horizontal distance. Therefore, we can apply this median principle along Streets and Avenues independently. You find that the preferred meeting point has coordinates which are the medians in the horizontal and the vertical directions, respectively.
Answer (iii): Location C.

(d) A fake gold coin got mixed up in a pile of 1000 real ones. All the 1001 coins look alike; from their visual appearance there is no telling which coin is real and which is fake. But the real coins all have the same weight, whereas the fake coin does not, though we don't know for sure if it is lighter or heavier.
At our disposal we have a traditional balance, by which we can compare the weight of two piles of coins placed at each of its scales, but which doesn't allow us to measure a numerical value.



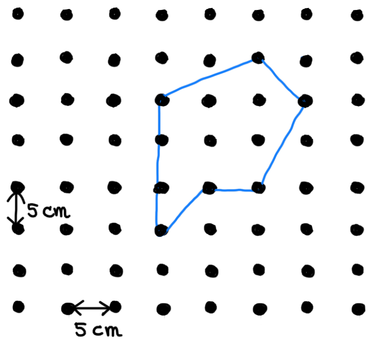
What is the **least number of weightings needed** to always find the fake coin and determine whether it is lighter or heavier than the real coins?

(i) 6
(ii) 7
(iii) 8
(iv) 9

SOLUTION (d)

Initially there are 2002 possible situations: coin 1 is heavier, coin 1 is lighter, coin 2 is heavier, coin 2 is lighter, and so on, until coin 1001.
Every time you use the balance to weigh, you will get one of three possible results: left is heavier, right is heavier, or the two sides weigh the same. The result is dependent on which coins you place where, and the actual situation at hand that you would like to learn about. The result of the first weighting divides the 2002 situations into three groups: those consistent with the first, second, and third possible outcomes. Likewise, every other weighting splits each of the earlier groups into three further groups. So, there's a factor 3 involved at every weighting. Now, 3^6 = 729 and 3^7 = 2187, showing that 2002 possible situations require at least 7 weightings. This can indeed be achieved (you will quickly find on the internet how to deal with 13 coins of which one is fake, in just three weightings. The same principle can be extended to larger numbers of coins.)
Answer (ii): 7 weightings.

(e) A fakir is playing with an elastic band on his bed of nails. These nails are arranged in a rectangular grid, with gaps of 5 cm horizontally and vertically between them. The fakir spans the elastic band around some of them. See the example below.

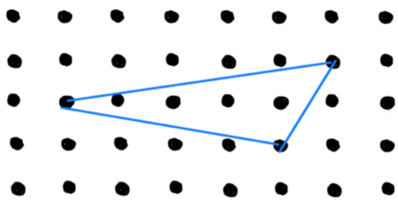


What is **the smallest area** that will always be enclosed when there are precisely 4 nails in the strict interior of the band (so, not counting those in the corners or along any edge)?
(i) 100 cm^2
(ii) 112.5 cm^2
(iii) 117.5 cm^2
(iv) 125 cm^2

SOLUTION (e)
To compute the area of a polygon having all its corners at grid points, there is a wonderful result: **Pick's theorem** helps you out. It states that the area A is easily computed by simply counting the number of interior grid points B and the number of grid points on the edges R (this includes corners). Pick's formula then reads as follows:

$A = B + R/2 – 1$

Now, in this question B=4 is fixed and to minimize A we want to minimize R. There will be at least 3 corner points on any polygon, so A = 4 + 3/2 – 1 = 4.5 is clearly the minimum, for which we must create a long triangle which has 4 interior points all in a row.



Since in our case the grid points are 5 cm apart in both directions, we get for the enclosed area 4.5 x 25 cm^2 = 112.5 cm^2.
Answer (ii): 112.5 cm^2.